

Exact solvability of the quantum Rabi models within Bogoliubov operators

Qing-Hu Chen^{1,2}, Chen Wang¹, Shu He^{1,3}, Tao Liu³, and Ke-Lin Wang⁴

¹ *Department of Physics, Zhejiang University, Hangzhou 310027, P. R. China*

² *Center for Statistical and Theoretical Condensed Matter Physics, Zhejiang Normal University, Jinhua 321004, P. R. China*

³ *School of Science, Southwest University of Science and Technology, Mianyang 621010, P. R. China*

⁴ *Department of Modern Physics, University of Science and Technology of China, Hefei 230026, P. R. China*

(Dated: March 2, 2013)

The quantum Rabi model can be solved exactly by the Bargmann transformation from real coordinate to complex variable recently [Phys. Rev. Lett. **107**, 100401 (2011)]. By the extended coherent states, we recover this solution in an alternative simpler and perhaps more physical way without uses of any extra conditions, like Bargmann conditions. In the same framework, the two-photon Rabi model are solved exactly by extended squeeze states. Transcendental functions have been derived with the similar form as those in one-photon model. Both extended coherent states and squeeze states are essentially Fock states in the space of the corresponding Bogoliubov operators. The present approach could be easily extended to study the exact solvability or integrability of various spin-boson systems with multi-level, even multi-mode.

PACS numbers: 03.65.Ge, 42.50.Pq 42.50.Lc

I. INTRODUCTION

Matter-light interaction is fundamental and ubiquitous in modern physics ranging from quantum optics, quantum information science to condensed matter physics. The simplest paradigm is a two-level atom (qubit) coupled to the electromagnetic mode of a cavity (oscillator), which is called Rabi model [1]. In the strong coupling regime where the coupling strength g/ω (ω is the cavity frequency) between the atom and the cavity mode exceeds the loss rates, the atom and the cavity can repeatedly exchange excitations before coherence is lost. The Rabi oscillations can be observed in this strong coupling atom-cavity system, which is usually called as cavity quantum electrodynamics (QED) [2]. Typically, the coupling strength in cavity QED reaches $g/\omega \sim 10^{-6}$. It can be described by the Jaynes-Cummings (JC) model [3] where the rotating-wave approximation (RWA) is made and analytically closed-form exact solutions are available.

Recently, for superconducting qubits, a one-dimensional (1D) transmission line resonator [4] or a LC circuit [5–8] can play a role of the cavity, which is known today as circuit QED. More recently, LC resonator inductively coupled to a superconducting qubit [9–11] has been realized experimentally. The qubit-resonator coupling has been strengthened to ten percentage. In this ultrastrong coupling regime of circuit QED, the evidence for the breakdown of the RWA has been provided by the transmission spectra [9]. The remarkable Bloch-Siegert shift associated with the counter-rotating terms also demonstrates the failure of the RWA [10]. So the quantum Rabi model (QRM) has been revisited by many authors.

In the representation of bosonic creation and annihilation operators in the Bargmann space [12] of analytical functions in a complex variable, Braak [13] recently derived a transcendental function in the QRM, which is

defined through power series in the coupling strength with coefficients related recursively. Zeros of transcendental functions can give exact eigenvalues. By a proposed criterion for quantum integrability, Braak further shows that the QRM is integrable due to the parity symmetry. However, the derivations seems to be outlined in a mathematical way. It was also suggested [14] that an intense dialogue between mathematics and physics would be needed. In other words, it is useful to shed some physical insights into Braak's mainly mathematical solutions.

In this paper, without the use of any extra conditions, like analyticity of the eigenfunction in Bargmann representation, we alternatively re-derive the same transcendental functions as in Ref. [13] quantum mechanically within the extended coherent states [15, 16]. Both zero bias and biased QRM can be treated simultaneously. Our method is more intuitionistic, and may be more easily understandable. The key procedures can be described with a simple tutorial, and therefore it is straightforward to extend to study the exact solvability or integrability of various spin-boson systems. The extension to the two-photon QRM [17, 18] is performed in this paper as the first example.

II. THE QRM WITHIN BOGOLIUBOV OPERATORS

A. Re-derivation of Braak's solution

The Hamiltonian of a generalized QRM can be describe as follows

$$H = -\frac{1}{2}(\varepsilon\sigma_z + \Delta\sigma_x) + a^\dagger a + g(a^\dagger + a)\sigma_z, \quad (1)$$

where ε and Δ are qubit static bias and tunneling matrix element, a^\dagger , a are the photon creation and annihilation

operators of the single-mode cavity with frequency ω , g is the qubit-cavity coupling constant, and $\sigma_k (k = x, y, z)$ are the Pauli matrices. To facilitate the study, we write the Hamiltonian in the matrix form in units of $\hbar = \omega = 1$

$$H = \begin{pmatrix} a^\dagger a + g(a^\dagger + a) - \frac{\varepsilon}{2} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g(a^\dagger + a) + \frac{\varepsilon}{2} \end{pmatrix}. \quad (2)$$

To remove the linear terms in $a^\dagger(a)$ operators, we perform the following two Bogoliubov transformations

$$A = a + g, B = a - g. \quad (3)$$

In Bogoliubov operators $A(B)$, the matrix element H_{11} (H_{22}) can be reduced to the free particle number operators $A^\dagger A$ ($B^\dagger B$) plus a constant, which is very helpful for the further study.

Different from the previous ansatz that the Hamiltonian is expressed in the two operators A and B at the same time[15], we here use the single operator one by one. First, in terms of operator A , the Hamiltonian can be written as

$$H = \begin{pmatrix} A^\dagger A - \alpha & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & A^\dagger A - 2g(A^\dagger + A) + \beta \end{pmatrix}, \quad (4)$$

where

$$\alpha = g^2 + \frac{\varepsilon}{2}, \quad \beta = 3g^2 + \frac{\varepsilon}{2}.$$

The wavefunction is then proposed as

$$|\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A \\ \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \end{pmatrix}, \quad (5)$$

where e_n and f_n are the expansion coefficients, $|n\rangle_A$ is called extended coherent state with the following properties

$$|n\rangle_A = \frac{(A^\dagger)^n}{\sqrt{n!}} |0\rangle_A = \frac{(a^\dagger + g)^n}{\sqrt{n!}} |0\rangle_A, \quad (6)$$

$$|0\rangle_A = e^{-\frac{1}{2}g^2 - ga^\dagger} |0\rangle_a. \quad (7)$$

The Schrödinger equation gives

$$\begin{aligned} & \sum_{n=0}^{\infty} (n - \alpha) \sqrt{n!} e_n |n\rangle_A - \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \\ &= E \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A \\ & - \frac{\Delta}{2} \sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A + \sum_{n=0}^{\infty} (n + \beta) \sqrt{n!} f_n |n\rangle_A \\ & - 2g \sum_{n=0}^{\infty} \left(\sqrt{n} f_n \sqrt{n!} |n-1\rangle_A + \sqrt{n+1} \sqrt{n!} f_n |n+1\rangle_A \right) \\ &= E \sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A \end{aligned}$$

Left multiplying ${}_A \langle m|$ gives

$$(m - \alpha - E) e_m = \frac{\Delta}{2} f_m, \quad (8)$$

$$(m + \beta - E) f_m - 2g(m+1) f_{m+1} - 2g f_{m-1} = \frac{\Delta}{2} e_m. \quad (9)$$

So the two coefficients e_n and f_n with the same index n are related with

$$e_m = \frac{\Delta}{2(m - \alpha - E)} f_m, \quad (10)$$

and the coefficient f_n can be defined recursively,

$$m f_m = \Omega(m-1) f_{m-1} - f_{m-2}, \quad (11)$$

$$\Omega(m) = \frac{1}{2g} \left((m + \beta - E) - \frac{\Delta^2}{4(m - \alpha - E)} \right), \quad (12)$$

with $f_0 = 1$ and $f_1 = \Omega(0)$

Similarly, by the second operator B , we can have the Hamiltonian as

$$H = \begin{pmatrix} B^\dagger B + 2g(B^\dagger + B) + \beta' & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & B^\dagger B - \alpha' \end{pmatrix}, \quad (13)$$

where

$$\alpha' = g^2 - \frac{\varepsilon}{2}, \quad \beta' = 3g^2 - \frac{\varepsilon}{2}.$$

The wavefunction can be also suggested in terms of B as

$$|\rangle = \begin{pmatrix} \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |n\rangle_B \\ \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |n\rangle_B \end{pmatrix}. \quad (14)$$

Proceed as before, the two coefficients f'_n and e'_n satisfy

$$-2g(f'_{m-1} + (m+1)f'_{m+1}) + (m + \beta' - E) f'_m - \frac{\Delta}{2} e'_m = 0,$$

$$-\frac{\Delta}{2} f'_m + (m - \alpha') e'_m = E e'_m,$$

then we have

$$e'_m = \frac{\frac{\Delta}{2}}{m - \alpha' - E} f'_m, \quad (15)$$

$$m f'_m = \Omega'(m-1) f'_{m-1} - f'_{m-2}, \quad (16)$$

$$\Omega'(m) = \frac{1}{2g} \left[(m + \beta' - E) - \frac{\Delta^2}{4(m - \alpha' - E)} \right], \quad (17)$$

with $f'_0 = 1$ and $f'_1 = \Omega'(0)$.

If both wavefunctions (5) and (14) are the true eigenfunction for the same eigenvalue E , they should be in

principle only different by a complex constant r if this eigenvalue is not degenerate

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n |n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n |n\rangle_B, \quad (18)$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n |n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n |n\rangle_B. \quad (19)$$

Left multiplying the original vacuum state $\langle 0|$ to the both side of the above equations yields

$$\sum_{n=0}^{\infty} \sqrt{n!} e_n \langle 0|n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} f'_n \langle 0|n\rangle_B, \quad (20)$$

$$\sum_{n=0}^{\infty} \sqrt{n!} f_n \langle 0|n\rangle_A = r \sum_{n=0}^{\infty} (-1)^n \sqrt{n!} e'_n \langle 0|n\rangle_B, \quad (21)$$

where

$$\sqrt{n!} \langle 0|n\rangle_A = (-1)^n \sqrt{n!} \langle 0|n\rangle_B = e^{-g^2/2} g^n. \quad (22)$$

To eliminate the ratio constant r , we have the following relation

$$\sum_{n=0}^{\infty} e_n g^n \sum_{n=0}^{\infty} e'_n g^n = \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n,$$

with the help of Eqs. (10) and (15), we arrive at

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{\Delta/2}{n - \alpha - E} f_n g^n \sum_{n=0}^{\infty} \frac{\Delta/2}{n - \alpha' - E} f'_n g^n \\ &= \sum_{n=0}^{\infty} f_n g^n \sum_{n=0}^{\infty} f'_n g^n. \end{aligned} \quad (23)$$

If set $f_n = K_n^-$, $f'_n = K_n^+$ and $E = x - g^2$, we can recover Braak's exact solution[13]

$$G_\epsilon(x) = \left(\frac{\Delta}{2}\right)^2 \bar{R}^+(x) \bar{R}^-(x) - R^+(x) R^-(x) = 0, \quad (24)$$

where

$$\begin{aligned} R^\pm(x) &= \sum_{n=0}^{\infty} K_n^\pm(x) g^n, \\ \bar{R}^\pm(x) &= \sum_{n=0}^{\infty} \frac{K_n^\pm(x)}{x - n \mp \frac{\epsilon}{2}} g^n. \end{aligned}$$

If $\epsilon = 0$, the above equation can be reduced to the following zero-bias case obviously[13]

$$G_0^\pm(x) = \sum_{n=0}^{\infty} f_n(x) \left(1 \mp \frac{\Delta/2}{x - n}\right) g^n = 0. \quad (25)$$

Therefore Braak's G-functions are completely re-derived in a very intuitionistic and concise way.

The G-functions can be written [19] in terms of so-called Heun functions [20]. However, the zeros of this Heun functions can not be given analytically, the numerical technique in the search for the zeros is still needed, so the cut-off for the summation should be unavoidable in the practical evaluation.

B. Comparisons and discussions

In the above derivation, the key point is proportionality of the two wavefunctions (5) and (14) with the same eigenvalue. Both Hilbert spaces in the two Bogoliubov operators are complete, if truncation is not done, the proportionality is justified naturally for non-degenerate states. On the other hand, the degenerate eigenstates are excluded in principle here. It naturally follows that the Juddian solutions [21, 22], which eigenvalue is doubly degenerate, are exceptional to Braak's solution. Interestingly, we do not need the extra condition of analyticity of the eigenfunction in Bargmann representation. In addition, the validity of present approach is independent of the parity symmetry. The parity symmetry would be contained self-consistently in the final G-functions if the system really has, e.g. $\epsilon = 0$.

Based on two Bogoliubov operator A and B , three present authors and one collaborator have used the following wavefunction to the Hamiltonian(1) to analyze the spectrum in the qubit-oscillator systems [c.f. Eq. (6) in Ref. [16]]

$$|\rangle = \left(\frac{\sum_{n=0}^N c_n |n\rangle_A}{\sum_{n=0}^N d_n |n\rangle_B} \right), \quad (26)$$

where N is the truncated number. The numerical exact diagonalization (ED) in the space of the two Bogoliubov operators have given the spectrum exactly. The coefficients c_n and d_n can be obtained also.

It is interesting to link coefficients in wavefunction (26) and those in wavefunctions (5) and (14) as

$$\begin{aligned} c_n &= \sqrt{n!} e_n, \\ d_n &= r (-1)^n \sqrt{n!} e'_n, \end{aligned}$$

although the former ones are obtained from ED and the later ones by the zeros of the G functions. It can be also confirmed numerically. For practical interest, there are perhaps no essential differences between our work and Braak's solution[13], except that the avenues to obtain the basically same coefficients are different. The latter is described in a mathematical way and is of more conceptual interest.

In the mathematical sense, we can not rule out the possibility that the ED give good results for small N and get worse for higher N before the practical evaluation, although we know empirically that it is generally not that case for large N . For the low order perturbation theory, it happens that the third-order perturbation theory would give worse results than the second-order one in some parameter regime for instance, it may be not that case in very high order perturbation theory. In the calculation, we really find that relative difference between the exact ones, which are easily obtained to any desired accuracy, and those for the cut off N decreases monotonically with increasing N , and the convergence can be arrived at easily. One may know that the Heun series converges

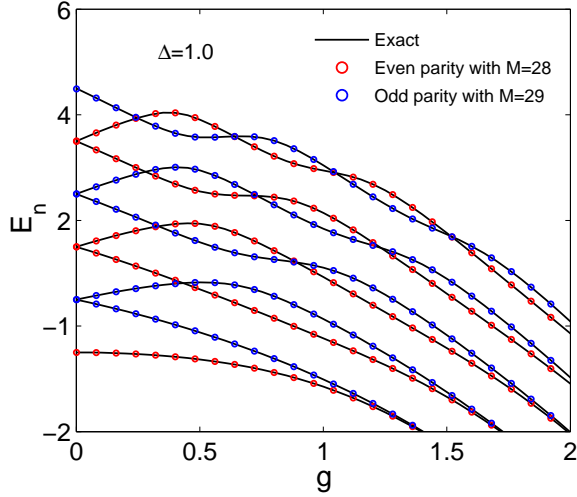


FIG. 1: (Color online) Spectrum of the one-photon quantum Rabi model as zeros of functions $F(\alpha)$ in Eq. (27) at resonance. The numerical solutions are also collected with solid lines.

before numerical calculations, although the cut-off can not be avoided in the real calculation. We indeed did not unfold the mathematics behind this formalism previously and focused on the analysis of the experimental spectrum data at that time[10].

Braak's G-functions exhibit a very compact form in power series, which motivate us to reshape our previous work. By tunable extended bosonic coherent states, the QRM can be mapped to a polynomial equation with a single variable[24]. We can also write this polynomial in power series as the following more concise form for large truncated number M

$$F(\alpha) = \sum_{j=0}^M \frac{(2\alpha)^j}{j!} c_{M-j} = 0, \quad (27)$$

where α is the key tunable variable we seek, and coefficients are also related recursively with the following scheme

$$(m+1)gc_{m+1} = -(m \pm \frac{\Delta}{2})c_m - (\alpha + g)c_{m-1} \pm (-1)^m \frac{\Delta}{2} \sum_{j=0}^m \frac{(2\alpha)^j}{j!} c_{m-j}, \quad (28)$$

initiated from $c_0 = 1.0$ and $c_1 = 0$, because the coefficients with the two highest indices M and $M-1$ are negligible small due to the required convergence and can thus be omitted. The zeros of the above function $F(\alpha)$ can also give the exact eigenvalue through

$$E^\pm = \alpha g \mp \frac{\Delta}{2}. \quad (29)$$

where \pm denotes the parity. The results are shown in Fig. 1. In Eq. (20) of the end of Ref. [24], we have

demonstrated that the wavefunction is equivalent to the expansion in the Fock space of displaced operators with tunable displacement.

It is very interesting to note that zeros of the both functions defined through different power series in Eqs. (25) and (27) can give the exact eigenvalues. In our practical evaluation, it is not more difficult to find the zeros for the function in Eq. (27) than those in Eq. (25), because the poles at $x = n$ emerging in the latter are not present in the former. The key difference between Eqs. (25) and (27) is that the former is well-defined without restriction and the latter is well-defined with build-in truncation.

It is implied in the viewpoint[14] that the QRM might have been solved exactly with an analytical closed-form solution in Ref. [13]. Nevertheless, whether Braak's exact solution could be called closed-form is subtle and therefore still controversial in our opinion. The so-called Heun functions can be basically called closed-form because they are well defined, although much more complicated than e.g. the hypergeometric functions. But the eigenvalues are given by the zeros of the Heun functions, which can not be obtained without truncation in the power series. As shown in wavefunctions (5) and (14), the expansion can not be closed naturally like in the JC model. It is generally accepted that the QRM has no trivial closed-form solution like that in the JC model due to the counter-rotating terms. The QRM can only have closed-form solutions with a vanishing qubit tunneling $\Delta = 0$ [24, 25]. Perhaps the question of "closed-form" solutions is academic and not of real value. Braak's solution is interesting for integrability of the QRM.

C. Tutorial for Bogoliubov operators approach

The present approach within the Bogoliubov operators can be generally described as follows, which is helpful for the further applications. The main task is to find the corresponding Bogoliubov operators. Then, one can expand the wavefunctions in terms of each Bogoliubov operator respectively. Eliminating the ratio constant of these wavefunctions will give the transcendental functions, which would be defined through power series in model parameter dependent quantities with coefficients related recursively. Finally, zeros of these transcendental functions would give the eigenvalues exactly, where numerical solutions to the one-variable (or finite variables in other multi-level systems for example) nonlinear equation must be required. Although the power series are defined through an infinite summation formally, in the practical calculation, it should be truncated to a finite summation. Fortunately, the obtained transcendental function $G(x)$ can be written in terms of so-called Heun functions, by which we know the convergence before the numerical solutions. The unavoidable "cut-off" in the summation of the G-functions in the practical calculations means that some states in the Hilbert space is not considered, according to the wavefunctions (5) and (14), even they con-

tribute negligible small, they still belong to the Hilbert space of Bogoliubov operators.

The applications to the two-photon QRM is performed in the next section as a preliminary example.

III. EXTENSIONS TO THE TWO-PHOTON QRM

The two-photon QRM is an phenomenological model describing a three-level system interacting with two photons. When the intermediate transition frequencies are strongly detuned from cavity frequency, after adiabatically eliminating the intermediate levels, one arrives at the effective Hamiltonian. This two-photon QRM has also been studied for a long time with the RWA[26] and beyond the RWA[17, 18, 27]. It may describe the two-photon processes occurring in Rubidium atoms[28] and quantum dots[29].

The Hamiltonian of two-photon QRM takes the following matrix form

$$H = \begin{pmatrix} a^\dagger a + g \left[(a^\dagger)^2 + a^2 \right] & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & a^\dagger a - g \left[(a^\dagger)^2 + a^2 \right] \end{pmatrix}. \quad (30)$$

First, we perform Bogoliubov transformation

$$b = ua + va^\dagger, b^\dagger = ua^\dagger + va, \quad (31)$$

to generate a new bosonic operator. Compared to the Hamiltonian, if set

$$u = \sqrt{\frac{\beta+1}{2}}, v = \sqrt{\frac{\beta-1}{2}}, \quad (32)$$

with $\beta = \frac{1}{\sqrt{1-4g^2}}$, we have a simple quadratic form of one diagonal Hamiltonian matrix element

$$H_{11} = a^\dagger a + g \left[(a^\dagger)^2 + a^2 \right] = \frac{b^\dagger b - v^2}{u^2 + v^2}.$$

Similarly, we can introduce another operator

$$c = ua - va^\dagger, c^\dagger = ua^\dagger - va, \quad (33)$$

which yields a simple quadratic form of the other diagonal Hamiltonian matrix element

$$H_{22} = a^\dagger a - g \left[(a^\dagger)^2 + a^2 \right] = \frac{c^\dagger c - v^2}{u^2 + v^2}.$$

In terms of the Bogoliubov operator b , the Hamiltonian can be written as

$$H = \begin{pmatrix} \frac{b^\dagger b - v^2}{u^2 + v^2} - v^2 & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & H'_{22} \end{pmatrix}, \quad (34)$$

with

$$H'_{22} = (u^2 + v^2 + 4guv) b^\dagger b - [uv + g(u^2 + v^2)] \left[(b^\dagger)^2 + b^2 \right] + 2guv + v^2.$$

The wavefuctions is suggested as

$$| \rangle = \begin{pmatrix} \sum_{n=0} \sqrt{n!} e_n |n\rangle_b \\ \sum_{n=0} \sqrt{n!} f_n |n\rangle_b \end{pmatrix}, \quad (35)$$

where

$$|n\rangle_b = \frac{(b^\dagger)^n}{\sqrt{n!}} |0\rangle_b = \frac{(ua^\dagger + va)^n}{\sqrt{n!}} |0\rangle_b, \quad (36)$$

$$b|0\rangle_b = 0, \quad (37)$$

$|0\rangle_b$ is simply the single-mode squeezed vacuum state, $|n\rangle_b$ is thus called as extended squeezed state. The Schrödinger equation gives

$$\begin{aligned} & \sum_{n=0} \frac{b^\dagger b - v^2}{u^2 + v^2} \sqrt{n!} e_n |n\rangle_b - \frac{\Delta}{2} \sum_{n=0} \sqrt{n!} f_n |n\rangle_b \\ &= E \sum_{n=0} \sqrt{n!} e_n |n\rangle_b, \\ & (u^2 + v^2 + 4guv) b^\dagger b \sum_{n=0} \sqrt{n!} f_n |n\rangle_b \\ & - [uv + g(u^2 + v^2)] \left[(b^\dagger)^2 + b^2 \right] \sum_{n=0} \sqrt{n!} f_n |n\rangle_b \\ & + (2guv + v^2) \sum_{n=0} \sqrt{n!} f_n |n\rangle_b - \frac{\Delta}{2} \sum_{n=0} \sqrt{n!} e_n |n\rangle_b \\ &= E \sum_{n=0} \sqrt{n!} f_n |n\rangle_b. \end{aligned}$$

Left multiplying ${}_b \langle m|$ gives

$$\begin{aligned} & \left(\frac{m - v^2}{u^2 + v^2} - E \right) e_m - \frac{\Delta}{2} f_m = 0, \\ & - [uv + g(u^2 + v^2)] [f_{m-2} + (m+2)(m+1)f_{m+2}] \\ & + [(u^2 + v^2)m + v^2 + 2guv(2m+1) - E] f_m \\ & - \frac{\Delta}{2} e_m = 0. \end{aligned}$$

Then we have build one-by-one relation for coefficient e_m and f_m

$$e_m = \frac{\Delta}{2 \left[\frac{m-v^2}{u^2+v^2} - E \right]} f_m, \quad (38)$$

which will considerably simplify the problem. The recursive relation is then obtained as

$$(m+2)(m+1)f_{m+2} = -f_{m-2} + \frac{\Omega(m)}{uv + g(u^2 + v^2)} f_m, \quad (39)$$

where

$$\begin{aligned} \Omega(m) &= (u^2 + v^2)m + v^2 + 2guv(2m+1), \\ & -E - \frac{\Delta^2}{4 \left(\frac{m-v^2}{u^2+v^2} - E \right)}. \end{aligned} \quad (40)$$

The Hamiltonian can also be expressed in terms of the other Bogoliubov operator c

$$H = \begin{pmatrix} H'_{11} & -\frac{\Delta}{2} \\ -\frac{\Delta}{2} & \frac{c^\dagger c - v^2}{u^2 + v^2} \end{pmatrix}, \quad (41)$$

with

$$H'_{11} = (v^2 + u^2 + 4guv) c^\dagger c + [uv + g(v^2 + u^2)] [(c^\dagger)^2 + c^2] + 2guv + v^2.$$

The wavefunction then can be expanded in the Fock space of the c operator as the following form

$$|\rangle = \left(\sum_{n=0} i^l \sqrt{n!} f'_n |n\rangle_c \right), \quad (42)$$

where $l = n$ for $n = 2k$ and $l = n + 1$ for $n = 2k + 1$. Therefore only two values of $i^l = \pm 1$ are possible.

Similarly, the Schrödinger equation gives the following relations

$$\begin{aligned} -\frac{\Delta}{2} f'_m + \frac{m - v^2}{u^2 + v^2} e'_m &= E e'_m, \\ -[g(v^2 + u^2) + uv] [(m+2)(m+1) f'_{m+2} + f'_{m-2}] \\ + [v^2 + (u^2 + v^2)m + 2guv(2m+1) - E] f'_m - \frac{\Delta}{2} e'_m &= 0. \end{aligned}$$

The coefficient e_m and f_m are related by

$$e'_m = \frac{\Delta}{2 \left[\frac{m - v^2}{u^2 + v^2} - E \right]} f'_m, \quad (43)$$

and the recursive relation is

$$(m+2)(m+1) f'_{m+2} = -f'_{m-2} + \frac{\Omega'(m)}{uv + g(u^2 + v^2)} f'_m, \quad (44)$$

with

$$\begin{aligned} \Omega'(m) &= v^2 + (u^2 + v^2)m + 2guv(2m+1) \\ &\quad - E - \frac{\Delta^2}{4 \left(\frac{m - v^2}{u^2 + v^2} - E \right)}. \end{aligned} \quad (45)$$

Note that the two sets of coefficients in the two wavefunctions has the same form.

Similarly, the two wavefunctions with the same eigenvalue should be in principle proportional with each other for the non-degenerate state

$$\left(\sum_{n=0} \sqrt{n!} e_n |n\rangle_b \right) = r \left(\sum_{n=0} i^l \sqrt{n!} f'_n |n\rangle_c \right). \quad (46)$$

Left multiplying $\langle 0|$ to the both equations gives

$$\begin{aligned} \sum_{n=0} \sqrt{n!} e_n \langle 0|n\rangle_b &= r \sum_{n=0} i^l \sqrt{n!} f'_n \langle 0|n\rangle_c, \\ \sum_{n=0} \sqrt{n!} f_n \langle 0|n\rangle_b &= r \sum_{n=0} i^l \sqrt{n!} e'_n \langle 0|n\rangle_c. \end{aligned}$$

We always have

$$i^l \sqrt{n!} \langle 0|n\rangle_c = \sqrt{n!} \langle 0|n\rangle_b = L_n^{e,o}, \quad (47)$$

where

$$L_{n=2k}^e = \frac{(2k)! (uv)^k}{2^k} \sum_{j=0}^k \frac{(-\frac{v^2}{u^2})^j}{j! (k-j)!}, \quad (48)$$

$$L_{n=2k+1}^o = \frac{(2k+1)! v (uv)^k}{2^k} \sum_{j=0}^k \frac{2^{2j} j! (-\frac{v^2}{u^2})^j}{(2j+1)! (k-j)!} \quad (49)$$

for even and odd particle numbers in the Bogoliubov operators b and c respectively. Then we have

$$\sum_n e_n L_n^{e,o} = r \sum_n f'_n L_n^{e,o}; \quad \sum_n f_n L_n^{e,o} = r \sum_n e'_n L_n^{e,o}.$$

Now the summation \sum_n is separated into two series with even and odd number n . To eliminate the constant r , we have

$$\sum_n \frac{\Delta}{2 \left(\frac{n - v^2}{u^2 + v^2} - E \right)} f_n L_n^{e,o} \sum_n \frac{\Delta}{2 \left(\frac{n - v^2}{u^2 + v^2} - E \right)} f'_n L_n^{e,o} = \sum_n f'_n L_n^{e,o} \sum_n f_n L_n^{e,o}, \quad (50)$$

with the use of Eqs. (38) and (43). Set $f_n = f'_n$ and $-x = -v^2 - E(u^2 + v^2)$, we finally have

$$G_{e,o}^\pm = \sum_n f_n \left[1 \pm \frac{\Delta(u^2 + v^2)}{2(n-x)} \right] L_n^{e,o} = 0, \quad (51)$$

where the coefficient f_n is initiated from $f_0 = 1$ ($f_1 = 1$) for the case of the even (odd) n in the recurrence scheme Eq. (39), and \pm denotes the parity. Thus, G-functions for the two-photon QRM have been obtained. The zeros of the G-functions give the exact eigenvalues, as shown in Fig. 2. It should be straightforward to extend to the biased two-photon QRM, which is not shown here.

Travěnek[18] has used Braak's approach to solve this model, but seems that the G-function like in Eq. (51) is not given. Their coefficients are entangled in the two coupled equations, which may prevent such a simple description for the G-functions. In the present Eqs. (38) and (43), two coefficients are related one-by-one with the same index n , which facilitates the derivations. This is also the advantage of Bogoliubov operators, which result in free particle number operators.

The exceptional solutions to the two-photon QRM have been studied by Emary et al [23]. With the G-function in the two-photon QRM at hand, we can also discuss the Juddian solution similar to the one-photon model[13]. The G function is also not analytic in x but has simple poles at $x = 0, 1, 2, \dots$. For special values of model parameters g , there are eigenvalues which do not

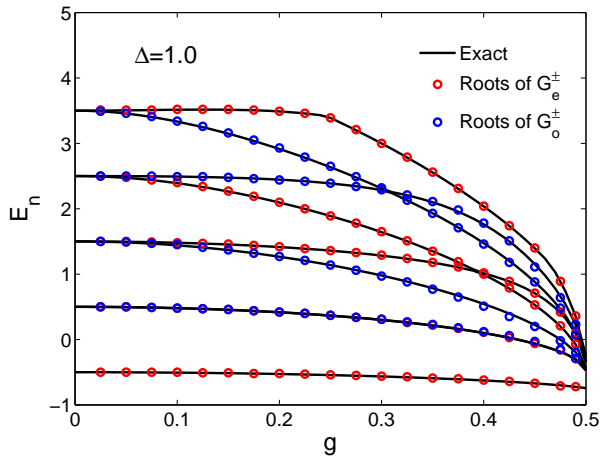


FIG. 2: (Color online) Spectrum of the two-photon quantum Rabi model as zeros of G-functions in Eq. (51). The numerical solutions are also collected with solid lines.

correspond to zeros of Eq. (51); these are the exceptional solutions. All exceptional eigenvalues are given by the positions of the poles

$$E = \left(n + \frac{1}{2}\right) \sqrt{1 - 4g^2} - \frac{1}{2} \quad (52)$$

which is exactly the same as that in Ref. [23]. The necessary and sufficient condition for the occurrence of the eigenvalue is $f_n(x = n) = 0$, which provides a condition on the model parameters g and Δ . They occur when the pole of $G_{e,o}^{\pm}(x)$ at $x = n$ is lifted because its numerator in Eq. (51) vanishes. From Eq. (50), we know the proportionality is justified only for the even or odd photonic number respectively. The Juddian solutions are corresponding to those states which are degenerate, and therefore are excluded within this proportionality, so the level crossing points with the same even, or odd photonic numbers are corresponding to Juddian solutions.

IV. SUMMARY

In this paper, by using the extended coherent states, Braak's exact solution in the QRM is recovered explicitly in an alternative more physical way. A preliminary extension to the two-photon QRM is also performed with the use of extended squeeze states. The corresponding G-functions with the similar form of Braak's G-function are derived, which has not been obtained before. Both model can be treated in an unified way by the expansion in the Fock space of the Bogoliubov operators. Further extensions to other more complicated systems, such as multi-level, even multi-mode spin-boson model, are straightforward, although perhaps a little bit tedious sometimes.

For multi-level spin-boson model, such as the finite-sized Dicke model[30], the quantum chaos has been discussed[31]. We have expanded the wavefunction in $N + 1$ Bogoliubov operators for Dicke model with finite N two-level atoms, and get numerically exact solutions previously[15]. According to the above discussions and the link with Braak's solutions, the exact solvability is ensured without doubt in this system. The quasi-integrability and the quantum chaos in this system should be very interesting. On the other hand, the multi-mode QRM has been also realized experimentally in circuit QED systems[9]. The extensions to these systems are in progressing.

V. ACKNOWLEDGEMENTS

We acknowledge useful discussions with Victor V. Albert, Daniel Braak, I. Travěnec, and Yu-Yu Zhang. This work was supported by National Natural Science Foundation of China under Grant No. 11174254, National Basic Research Program of China (Grant Nos. 2011CBA00103 and 2009CB929104).

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